INTEGRAL GROUP RING OF THE MATHIEU SIMPLE GROUP M_{24}

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Dedicated to 70th birthday of Professor L.A. Bokut

Abstract. We consider the Zassenhaus conjecture for the normalized unit group of the integral group ring of the Mathieu sporadic group M_{24}. As a consequence, for this group we confirm Kimmerle's conjecture on prime graphs.

1. Introduction, conjectures and main results

Let $V(\mathbb{Z}G)$ be the normalized units group of the integral group ring $\mathbb{Z}G$ of a finite group G. A famous Zassenhaus conjecture [25] says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G.

For finite simple groups, the main tool for the investigation of the Zassenhaus conjecture is the Luthar–Passi method, introduced in [21] to solve this conjecture for A_5. Later M. Hertweck improved this method in [16] and used it for the investigation of $PSL(2, F_{p^n})$. The Luthar–Passi method proved to be useful for groups containing non-trivial normal subgroups as well. Also some related properties and some weakened variations of the Zassenhaus conjecture as well can be found in [1, 22] and [3, 20]. For some recent results we refer to [5, 7, 15, 16, 17, 18].

First of all, we need to introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G. The following weakened variation of the Zassenhaus conjecture was proposed in [20]:

Conjecture 1. (KC) *If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

In particular, in the same paper W. Kimmerle verified that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to **(ZC)** the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 8, 9, 11] **(KC)** was confirmed for the Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23} and the sporadic Janko simple groups J_1, J_2 and J_3.

Here we continue these investigations for the Mathieu simple group M_{24}. Despite using the Luthar–Passi method we are able to prove the rationally conjugacy only for torsion units of order 23 in $V(\mathbb{Z}M_{24})$, our main result gives a lot of information on partial augmentations of possible torsion units and allows us to confirm **(KC)** for the sporadic group M_{24}.

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It is well-known that the collection of conjugacy classes of M_{24} is

$$\begin{aligned}\mathcal{C} = \{\, C_1,\ C_{2a}, C_{2b},\ C_{3a},\ C_{3b},\ C_{4a},\ C_{4b},\ C_{4c},\\ C_{5a}, C_{6a},\ C_{6b},\ C_{7a},\ C_{7b},\ C_{8a},\ C_{10a},\ C_{11a},\ C_{12a},\\ C_{12b}, C_{14a},\ C_{14b},\ C_{15a},\ C_{15b},\ C_{21a},\ C_{21b},\ C_{23a},\ C_{23b}\,\},\end{aligned}$$

where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g\in C_{nt}} \alpha_g$, the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [2] and [24], Ch.5, p.102) one knows that $\nu_1 = \alpha_1 = 0$ and

$$\sum_{C_{nt}\in\mathcal{C}} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt}\chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}.

The main result is the following.

Theorem 1. *Let G denote the Mathieu simple group M_{24}. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ and let*

$$\begin{aligned}\mathfrak{P}(u) = (\nu_{2a}, \nu_{2b},\ \nu_{3a},\ \nu_{3b},\ \nu_{4a},\ \nu_{4b},\ \nu_{4c},\ \nu_{5a},\ \nu_{6a},\ \nu_{6b},\ \nu_{7a},\ \nu_{7b},\ \nu_{8a},\ \nu_{10a},\\ \nu_{11a},\ \nu_{12a},\ \nu_{12b},\ \nu_{14a},\ \nu_{14b},\ \nu_{15a},\ \nu_{15b},\ \nu_{21a},\ \nu_{21b},\ \nu_{23a},\ \nu_{23b}\,) \in \mathbb{Z}^{25}\end{aligned}$$

be the tuple of partial augmentations of u. The following properties hold.

- (i) *There is no elements of orders 22, 33, 35, 46, 55, 69, 77, 115, 161 and 253 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{20, 24, 28, 30, 40, 42, 56, 60, 84, 120, 168\}$, then $|u|$ coincides with the order of some element $g \in G$.*
- (ii) *If $|u| \in \{5, 11, 23\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 2$, the tuple of the partial augmentations of u belongs to the set*

$$\begin{aligned}\big\{\ (\mathfrak{P}(u) \in \mathbb{Z}^{25} \mid (\nu_{2a}, \nu_{2b}) \in \{\ (0,1),\ (-2,3),\ (2,-1),\ (1,0),\\ (3,-2),\ (-1,2)\},\quad \nu_{kx} = 0,\ kx \notin \{2a, 2b\}\ \big\}.\end{aligned}$$

- (iv) *If $|u| = 3$, the tuple of the partial augmentations of u belongs to the set*

$$\begin{aligned}\big\{\ (\mathfrak{P}(u) \in \mathbb{Z}^{25} \mid (\nu_{3a}, \nu_{3b}) \in \{\ (0,1),\ (2,-1),\ (1,0),\ (3,-2),\\ (-1,2),\ (4,-3)\ \},\quad \nu_{kx} = 0,\ kx \notin \{3a, 3b\}\ \big\}.\end{aligned}$$

- (v) *If $|u| = 7$, the tuple of the partial augmentations of u belongs to the set*

$$\begin{aligned}\big\{\ (\mathfrak{P}(u) \in \mathbb{Z}^{25} \mid (\nu_{7a}, \nu_{7b}) \in \{\ (0,1),\ (2,-1),\ (1,0),\ (-1,2)\ \},\\ \nu_{kx} = 0,\ kx \notin \{7a, 7b\}\ \big\}.\end{aligned}$$

- (vi) *If $|u| = 10$, the tuple of the partial augmentations of u belongs to the set*

$$\begin{aligned}\big\{\quad (\mathfrak{P}(u) \in \mathbb{Z}^{25} \mid (\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}) \in \{\ (-3,1,5,-2), (-2,0,5,-2),\\ (-2,2,5,-4), (-1,-1,5,-2), (-1,1,5,-4), (0,-2,0,3), (0,0,0,1),\\ (0,2,0,-1), (1,-1,0,1), (1,1,0,-1), (1,3,0,-3)\ \},\\ \nu_{kx} = 0,\ kx \notin \{\nu_{2a}, \nu_{2b}, \nu_{5a}, \nu_{10a}\}\ \big\}.\end{aligned}$$

Note that using our implementation of the Luthar–Passi method, which we intend to make available in the GAP package LAGUNA [10], it is possible to compute 34 possible tuples of partial augmentations for units of order 15 and 21 tuple for units of order 21 listed in the Appendix.

As an immediate consequence of the part (i) of the Theorem we obtain

Corollary 1. *If $G = M_{24}$ then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. Preliminaries

The following result relates the solution of the Zassenhaus conjecture to partial augmentations of torsion units.

Proposition 1. *(see [21] and Theorem 2.5 in [23]) Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

The next result already yield that several partial augmentations are zero.

Proposition 2. *(see [15], Proposition 3.1; [16], Proposition 2.2) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3. *(see [16, 21]) Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G then, for every integer l, the number*

$$\mu_l(u, \chi, p) = \tfrac{1}{k} \textstyle\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\} \qquad (2)$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4. *(see [12]) The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

3. Proof of the Theorem

Throughout this section we denote M_{24} by G. It is well known [14] that $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $exp(G) = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. The character table of G, as well as the p-Brauer character tables, where $p \in \{2, 3, 5, 7, 11, 23\}$, can be found using the computational algebra system GAP [14], which derives these data from [13, 19]. Throughout the paper we will use the notation, inclusive the indexation, for the characters and conjugacy classes as used in the GAP Character Table Library.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14, 15, 21 and 23, first of all we will investigate units of some of these orders (except units of orders 4, 6, 8, 12 and 14). After this, by Proposition 4, the order of each torsion unit divides the exponent of G, and in the first instance we should consider units of orders 20, 22, 24, 28, 30, 33, 35, 42, 46, 55, 69, 77, 115, 161 and 253. We

will omit orders 20, 24, 28, 30 and 42 that do not contribute to **(KC)**, and this enforces us to add to the list of exceptions in part (i) of Theorem also orders 40, 56, 60, 84, 120 and 168, but no more because of restrictions imposed by the exponent of G.

Thus, we will prove that units of orders 22, 33, 35, 46, 55, 69, 77, 115, 161 and 253 do not appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

• Let u be an involution. By (1) and Proposition 2 we have that $\nu_{2a} + \nu_{2b} = 1$. Applying Proposition 3 to characters χ_2 we get the following system

$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(7\nu_{2a} - \nu_{2b} + 23) \geq 0; \quad \mu_1(u, \chi_2, *) = \tfrac{1}{2}(-7\nu_{2a} + \nu_{2b} + 23) \geq 0.$$

From these restrictions and the requirement that all $\mu_i(u, \chi_j, p)$ must be non-negative integers we get the six pairs (ν_{2a}, ν_{2b}) listed in part (iii) of our Theorem.

• Let u be a unit of order 3. By (1) and Proposition 2 we get $\nu_{3a} + \nu_{3b} = 1$. By (2) we obtain the system of inequalities

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \tfrac{1}{3}(10\nu_{3a} - 2\nu_{3b} + 23) \geq 0; \\ \mu_1(u, \chi_2, *) &= \tfrac{1}{3}(-5\nu_{3a} + \nu_{3b} + 23) \geq 0. \end{aligned}$$

Clearly, using the condition for $\mu_i(u, \chi_j, p)$ to be non-negative integers, we obtain six pairs (ν_{3a}, ν_{3b}) listed in part (iv) of the Theorem 1.

• Let u be a unit of order either 5 or 11. Using Proposition 2 and (2) we obtain that all partial augmentations except one are zero. Thus by Proposition 2 the particular proof of part (ii) of the Theorem 1 is done.

• Let u be a unit of order 7. By (1) and Proposition 2 we get $\nu_{7a} + \nu_{7b} = 1$. By (2) we obtain the system of inequalities

$$\begin{aligned} \mu_1(u, \chi_3, *) &= \tfrac{1}{7}(4\nu_{7a} - 3\nu_{7b} + 45) \geq 0; \\ \mu_1(u, \chi_2, 2) &= \tfrac{1}{7}(-4\nu_{7a} + 3\nu_{7b} + 11) \geq 0. \end{aligned}$$

Again, using the condition for $\mu_i(u, \chi_j, p)$ to be non-negative integers, we obtain four pairs (ν_{7a}, ν_{7b}) listed in part (v) of the Theorem 1.

• Let u be a unit of order 10. By (1) and Proposition 2 we have that

$$\nu_{2a} + \nu_{2b} + \nu_{5a} + \nu_{10a} = 1. \tag{3}$$

Since $|u^5| = 2$, we need to consider six cases defined by part (iii) of the Theorem 1.
Case 1. Let $\chi(u^5) = \chi(2a)$. Put

$$\begin{aligned} &t_1 = 7\nu_{2a} - \nu_{2b} + 3\nu_{5a} - \nu_{10a}, \qquad t_2 = 3\nu_{2a} - 5\nu_{2b}, \\ &t_3 = 14\nu_{2a} + 6\nu_{2b} + \nu_{5a} + \nu_{10a}. \end{aligned} \tag{4}$$

Applying Proposition 3, we get the system with indeterminates t_1, t_2 and t_3

$$\begin{aligned} \mu_0(u, \chi_2, *) &= \tfrac{1}{10}(4t_1 + 42) \geq 0; \quad \mu_5(u, \chi_2, *) = \tfrac{1}{10}(-4t_1 + 28) \geq 0; \\ \mu_5(u, \chi_3, *) &= \tfrac{1}{10}(4t_2 + 48) \geq 0; \quad \mu_0(u, \chi_3, *) = \tfrac{1}{10}(-12t_2 + 42) \geq 0; \\ \mu_0(u, \chi_7, *) &= \tfrac{1}{10}(8t_3 + 288) \geq 0; \quad \mu_5(u, \chi_7, *) = \tfrac{1}{10}(-8t_3 + 232) \geq 0. \end{aligned}$$

Its solution are $t_1 \in \{-8, -3, 2, 7\}$, $t_2 \in \{-12, -7, -2, 3, 8\}$ and

$$t_3 \in \{-36, -31, -26, -21, -16, -11, -6, -1, 4, 9, 14, 19, 24, 29\}.$$

Substituting values of t_1, t_2 and t_3 in (4), and adding the condition (3), we obtain the system of linear equations for ν_{2a}, ν_{2b}, ν_{5a}, and ν_{10a}. Since $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 7 & -1 & 3 & -1 \\ 3 & -5 & 0 & 0 \\ 14 & 6 & 1 & 1 \end{vmatrix} \neq 0$, this system has the unique solution for each t_1, t_2, t_3, and the only integer solutions are $(1,-1,0,1)$, $(1,1,0,-1)$ and $(1,3,0,-3)$.

Case 2. Let $\chi(u^5) = \chi(2b)$. Put $t_1 = 7\nu_{2a} - \nu_{2b} + 3\nu_{5a} - \nu_{10a}$, $t_2 = 3\nu_{2a} - 5\nu_{2b}$ and $t_3 = 14\nu_{2a} + 6\nu_{2b} + \nu_{5a} + \nu_{10a}$. Again using Proposition 3, we obtain that

$$\begin{aligned} \mu_0(u,\chi_2,*) &= \tfrac{1}{10}(4t_1+34) \geq 0; \quad & \mu_5(u,\chi_2,*) &= \tfrac{1}{10}(-4t_1+36) \geq 0; \\ \mu_5(u,\chi_3,*) &= \tfrac{1}{10}(4t_2+40) \geq 0; \quad & \mu_0(u,\chi_3,*) &= \tfrac{1}{10}(-4t_2+50) \geq 0; \\ \mu_0(u,\chi_7,*) &= \tfrac{1}{10}(8t_3+272) \geq 0; \quad & \mu_5(u,\chi_7,*) &= \tfrac{1}{10}(-8t_3+248) \geq 0. \end{aligned}$$

From this follows that $t_1 \in \{-6,-1,4,9\}$, $t_2 \in \{-10,-5,0,5,10\}$ and

$$t_3 \in \{-34,-29,-24,-19,-14,-9,-4,1,6,11,16,21,26,31\}.$$

Using the same considerations as in the previous case, we obtain only three solutions $(0,-2,0,3)$, $(0,0,0,1)$ and $(0,2,0,-1)$ that satisfy these restrictions and the condition that $\mu_i(u,\chi_j,p)$ are non-negative integers.

Case 3. Let $\chi(u^5) = -2\chi(2a) + 3\chi(2b)$. Put $t_1 = 7\nu_{2a} - \nu_{2b} + 3\nu_{5a} - \nu_{10a}$, $t_2 = 3\nu_{2a} - 5\nu_{2b}$ and $t_3 = 14\nu_{2a} + 6\nu_{2b} + \nu_{5a} + \nu_{10a}$. As before, by Proposition 3, we obtain that

$$\begin{aligned} \mu_0(u,\chi_2,*) &= \tfrac{1}{10}(4t_1+18) \geq 0; \quad & \mu_2(u,\chi_2,*) &= \tfrac{1}{10}(-t_1+3) \geq 0; \\ \mu_5(u,\chi_3,*) &= \tfrac{1}{10}(4t_2+24) \geq 0; \quad & \mu_0(u,\chi_3,*) &= \tfrac{1}{10}(-tt_2+66) \geq 0; \\ \mu_0(u,\chi_7,*) &= \tfrac{1}{10}(8t_3+240) \geq 0; \quad & \mu_5(u,\chi_7,*) &= \tfrac{1}{10}(-8t_3+280) \geq 0. \end{aligned}$$

From the last system of inequalities, we get $t_1 = 3$, $t_2 \in \{-6,-1,4,9,14\}$ and

$$t_3 \in \{-30,-25,-20,-15,-10,-5,0,5,10,15,20,25,30,35\},$$

and using the same considerations as in the previous case, we deduce that there is only one solution $(-2,0,5,-2)$ satisfying the previous restrictions and the condition that $\mu_i(u,\chi_j,p)$ are non-negative integers.

Case 4. Let $\chi(u^5) = 2\chi(2a) - \chi(2b)$. Again, for the same t_1, t_2 and t_3 we have

$$\begin{aligned} \mu_1(u,\chi_2,*) &= \tfrac{1}{10}(t_1+5) \geq 0; \quad & \mu_5(u,\chi_2,*) &= \tfrac{1}{10}(-4t_1+20) \geq 0; \\ \mu_5(u,\chi_3,*) &= \tfrac{1}{10}(4t_2+56) \geq 0; \quad & \mu_0(u,\chi_3,*) &= \tfrac{1}{10}(-4t_2+34) \geq 0; \\ \mu_0(u,\chi_7,*) &= \tfrac{1}{10}(8t_3+304) \geq 0; \quad & \mu_5(u,\chi_7,*) &= \tfrac{1}{10}(-8t_3+216) \geq 0. \end{aligned}$$

It follows that $t_1 \in \{-5,5\}$, $t_2 \in \{-14,-9,-4,1,6\}$ and

$$t_3 \in \{-38,-33,-28,-23,-18,-13,-8,-3,2,7,12,17,22,27\},$$

and we obtain tree solutions $\{\, (-3,0,5,-1),\ (-3,1,5,-2),\ (2,0,-5,4) \,\}$ satisfying the inequalities above. Now using the following additional inequalities:

$$\begin{aligned} \mu_1(u,\chi_3,*) &= \tfrac{1}{10}(-3\nu_{2a} + 5\nu_{2b} + 56) \geq 0; \\ \mu_5(u,\chi_5,11) &= \tfrac{1}{10}(-84\nu_{2a} - 52\nu_{2b} + 4\nu_{5a} - 12\nu_{10a} + 196) \geq 0, \end{aligned}$$

it remains only one solution $(-3,1,5,-2)$.

Case 5. Let $\chi(u^5) = 3\chi(2a) - 2\chi(2b)$. Put $t_1 = 7\nu_{2a} - \nu_{2b} + 3\nu_{5a} - \nu_{10a}$, $t_2 = 3\nu_{2a} - 5\nu_{2b}$ and $t_3 = 14\nu_{2a} + 6\nu_{2b} + \nu_{5a} + \nu_{10a}$. Again by (2) we obtain that

$$\mu_1(u,\chi_2,*) = \tfrac{1}{10}(t_1 - 3) \geq 0; \qquad \mu_5(u,\chi_2,*) = \tfrac{1}{10}(-4t_1 + 12) \geq 0;$$
$$\mu_5(u,\chi_3,*) = \tfrac{1}{10}(4t_2 + 64) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{10}(-4t_2 + 26) \geq 0;$$
$$\mu_0(u,\chi_7,*) = \tfrac{1}{10}(8t_3 + 320) \geq 0; \quad \mu_5(u,\chi_7,*) = \tfrac{1}{10}(-8t_3 + 200) \geq 0.$$

It is easy to check that $t_1 = 3$, $t_2 \in \{-16, -11, -6, -1, 4\}$ and

$$t_3 \in \{-40, -35, -30, -25, -20, -15, -10, -5, 0, 5, 10, 15, 20, 25\}.$$

So we obtained the following five solutions:

$$\{(-2,-2,5,0), (-2,-1,5,-1), (-2,0,5,-2), (-2,1,5,-3), (-2,2,5,-4)\}.$$

Now after using the following two additional inequalities:

$$\mu_1(u,\chi_3,*) = \tfrac{1}{10}(-3\nu_{2a} + 5\nu_{2b} + 64) \geq 0;$$
$$\mu_0(u,\chi_5,11) = \tfrac{1}{10}(84\nu_{2a} + 52\nu_{2b} - 4\nu_{5a} + 12\nu_{10a} + 262) \geq 0,$$

it remains only two solutions $\{\ (-2,0,5,-2),\ (-2,2,5,-4)\ \}$.
Case 6. Let $\chi(u^5) = -\chi(2a) + 2\chi(2b)$. Put $t_1 = 7\nu_{2a} - \nu_{2b} + 3\nu_{5a} - \nu_{10a}$, $t_2 = 3\nu_{2a} - 5\nu_{2b}$ and $t_3 = 14\nu_{2a} + 6\nu_{2b} + \nu_{5a} + \nu_{10a}$. Similarly, we get

$$\mu_0(u,\chi_2,*) = \tfrac{1}{10}(4t_1 + 26) \geq 0; \quad \mu_2(u,\chi_2,*) = \tfrac{1}{10}(-t_1 + 11) \geq 0;$$
$$\mu_5(u,\chi_3,*) = \tfrac{1}{10}(4t_2 + 32) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{10}(-4t_2 + 58) \geq 0;$$
$$\mu_0(u,\chi_7,*) = \tfrac{1}{10}(8t_3 + 256) \geq 0; \quad \mu_5(u,\chi_7,*) = \tfrac{1}{10}(-8t_3 + 264) \geq 0.$$

We have the following restrictions: $t_1 \in \{1, 11\}$, $t_2 \in \{-8, -3, 2, 7, 12\}$ and

$$t_3 \in \{-32, -27, -22, -17, -12, -7, -2, 3, 8, 13, 18, 23, 28, 33\},$$

that lead to the following five solutions

$$\{(-1,-3,5,0),\ (-1,-2,5,-1),\ (-1,-1,5,-2),\ (-1,0,5,-3),\ (-1,1,5,-4)\}$$

which satisfy the above inequalities. After considering two additional inequalities

$$\mu_1(u,\chi_3,*) = \tfrac{1}{10}(-3\nu_{2a} + 5\nu_{2b} + 32) \geq 0;$$
$$\mu_0(u,\chi_5,11) = \tfrac{1}{10}(84\nu_{2a} + 52\nu_{2b} - 4\nu_{5a} + 12\nu_{10a} + 230) \geq 0,$$

only two solutions remains: $\{\ (-1,-1,5,-2),\ (-1,1,5,-4)\ \}$.

Thus, the union of solutions for all six cases gives us part (vi) of the Theorem.

• Let u be a unit of order 15. By (1) and Proposition 2 we obtain that

$$\nu_{3a} + \nu_{3b} + \nu_{5a} + \nu_{15a} + \nu_{15b} = 1.$$

Since $|u^5| = 3$, according to part (iv) of the Theorem we need to consider six cases. Using the LAGUNA package [10], in all of them we constructed and solved systems of inequalities that give us 34 solutions listed in the Appendix.

• Let u be a unit of order 21. By (1) and Proposition 2 we obtain that

$$\nu_{3a} + \nu_{3b} + \nu_{7a} + \nu_{7b} + \nu_{21a} + \nu_{21b} = 1.$$

We need to consider 24 cases determined by parts (iv) and (v) of the Theorem 1. We write down explicitly the details of the first case, the treatment of the other ones are similar. Our computation was helped by the LAGUNA package [10].

Let $\chi(u^3)=\chi(7a)$ and $\chi(u^7)=\chi(3a)$, for any character χ of G. Put

$$t_1=5\nu_{3a}-\nu_{3b}+2\nu_{7a}+2\nu_{7b}-\nu_{21a}-\nu_{21b},$$
$$t_2=6\nu_{3b}-\nu_{7a}-\nu_{7b}-\nu_{21a}-\nu_{21b},\quad t_3=3\nu_{3b}+3\nu_{7a}-4\nu_{7b}+3\nu_{21a}-4\nu_{21b},$$
$$t_4=\nu_{3a},\quad t_5=3\nu_{3b}-6\nu_{7a}+8\nu_{7b}+3\nu_{21a}-4\nu_{21b}.$$

Applying Proposition 3 to characters χ_2, χ_3, χ_4, χ_7 and χ_{15} we get

$$\begin{aligned}
\mu_0(u,\chi_2,*)&=\tfrac{1}{21}(5t_1+45)\geq 0; & \mu_7(u,\chi_2,*)&=\tfrac{1}{21}(-6t_1+30)\geq 0;\\
\mu_0(u,\chi_3,*)&=\tfrac{1}{21}(6t_2+42)\geq 0; & \mu_7(u,\chi_3,*)&=\tfrac{1}{21}(-3t_2+42)\geq 0;\\
\mu_1(u,\chi_3,*)&=\tfrac{1}{21}(t_3+49)\geq 0; & \mu_9(u,\chi_3,*)&=\tfrac{1}{21}(-2t_3+49)\geq 0;\\
\mu_0(u,\chi_7,*)&=\tfrac{1}{21}(108t_4+270)\geq 0; & \mu_7(u,\chi_7,*)&=\tfrac{1}{21}(-54t_4+243)\geq 0;\\
\mu_9(u,\chi_{15},*)&=\tfrac{1}{21}(2t_5+1043)\geq 0; & \mu_1(u,\chi_{15},*)&=\tfrac{1}{21}(-t_5+1043)\geq 0.
\end{aligned}$$

Solution of this system of inequalities gives $t_1\in\{-2,5\}$, $t_2\in\{-7,0,7,14\}$, $t_3\in\{-49,-28,-7,14\}$, $t_4=1$ and $t_5\in\{14+21k\mid -25\leq k\leq 49\}$.

Using computer we get 1200 solutions satisfying inequalities above.

After considering the following four additional inequalities

$$\begin{aligned}
&\mu_9(u,\chi_2,2)=\tfrac{1}{21}(-4\nu_{3a}+2\nu_{3b}+6\nu_{7a}-8\nu_{7b}-12\nu_{21a}+16\nu_{21b}+11)\geq 0;\\
&\mu_1(u,\chi_2,2)=\tfrac{1}{21}(2\nu_{3a}-\nu_{3b}-3\nu_{7a}+4\nu_{7b}+6\nu_{21a}-8\nu_{21b}+5)\geq 0;\\
&\mu_0(u,\chi_4,2)=\tfrac{1}{21}(-12\nu_{3a}+24\nu_{3b}-18\nu_{7a}-18\nu_{7b}-18\nu_{21a}-18\nu_{21b}+33)\geq 0;\\
&\mu_3(u,\chi_3,*)=\tfrac{1}{21}(-6\nu_{3b}+8\nu_{7a}-6\nu_{7b}+8\nu_{21a}-6\nu_{21b}+42)\geq 0,
\end{aligned}$$

it remains only two solutions: $\{\,(1,2,-1,1,-2,0),\ (1,2,1,-1,-1,-1)\,\}$.

Similarly, using the LAGUNA package [10] we can construct the system of inequalities for the remaining 23 cases. The union of all solutions give us the list of solutions given in the Appendix.

• Let u be a unit of order 23. By (1) and Proposition 2 we get $\nu_{23a}+\nu_{23b}=1$. By (2) we obtain the following system of inequalities

$$\begin{aligned}
\mu_1(u,\chi_{10},*)&=\tfrac{1}{23}(12\nu_{23a}-11\nu_{23b}+770)\geq 0;\\
\mu_5(u,\chi_{10},*)&=\tfrac{1}{23}(-11\nu_{23a}+12\nu_{23b}+770)\geq 0;\\
\mu_1(u,\chi_2,2)&=\tfrac{1}{23}(12\nu_{23a}-11\nu_{23b}+11)\geq 0;\\
\mu_5(u,\chi_2,2)&=\tfrac{1}{23}(-11\nu_{23a}+12\nu_{23b}+11)\geq 0;\\
\mu_1(u,\chi_7,2)&=\tfrac{1}{23}(-13\nu_{23a}+10\nu_{23b}+220)\geq 0;\\
\mu_5(u,\chi_7,2)&=\tfrac{1}{23}(10\nu_{23a}-13\nu_{23b}+220)\geq 0;\\
\mu_1(u,\chi_{10},2)&=\tfrac{1}{23}(25\nu_{23a}-21\nu_{23b}+320)\geq 0;\\
\mu_5(u,\chi_{10},2)&=\tfrac{1}{23}(-21\nu_{23a}+25\nu_{23b}+320)\geq 0,
\end{aligned}$$

which has only two trivial solutions $(\nu_{23a},\nu_{23b})\in\{(1,0),(0,1)\}$. Thus, by Proposition 1 we conclude that each torsion unit of order 23 is rationally conjugate to some $g\in G$, and this completes the proof of part (ii) of the Theorem.

• Let u be a unit of order 22. By (1) and Proposition 2 we have that

$$\nu_{2a}+\nu_{2b}+\nu_{11a}=1.$$

Since $|u^{11}| = 2$, we need to consider six cases for any character χ of G. They are defined by part (iii) of the Theorem. Put

$$(\alpha,\beta,\gamma,\delta) = \begin{cases} (40,26,58,52), & \text{if} \quad \chi(u^{11}) = \chi(2a); \\ (32,34,50,60), & \text{if} \quad \chi(u^{11}) = \chi(2b); \\ (16,5,34,76), & \text{if} \quad \chi(u^{11}) = -2\chi(2a)+\chi(2b); \\ (48,18,66,44), & \text{if} \quad \chi(u^{11}) = 2\chi(2a)-\chi(2b); \\ (-1,10,74,36), & \text{if} \quad \chi(u^{11}) = 3\chi(2a)-2\chi(2b); \\ (24,42,42,68), & \text{if} \quad \chi(u^{11}) = -\chi(2a)+2\chi(2b), \end{cases} \tag{5}$$

$$t_1 = 7\nu_{2a} - \nu_{2b} + \nu_{11a} \qquad \text{and} \qquad t_2 = 3\nu_{2a} - 5\nu_{2b} - \nu_{11a}. \tag{6}$$

If $\chi(u^{11}) = 3\chi(2a) - 2\chi(2b)$, by (2) we obtain the system

$$\begin{aligned} \mu_0(u,\chi_2,*) = \tfrac{1}{22}(10t_1+\alpha) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \tfrac{1}{22}(-10t_1+\beta) \geq 0; \\ \mu_{11}(u,\chi_3,*) = \tfrac{1}{22}(10t_2+\gamma) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{22}(-10t_2+\delta) \geq 0. \end{aligned} \tag{7}$$

For each of the cases of (7), we solve the system (6) for t_1 and t_2. Then we obtain the following six solutions.

(i) $\chi(u^{11}) = \chi(2a)$. We get $t_1 = 4$ and $t_2 = 3$.
(ii) $\chi(u^{11}) = \chi(2b)$. We get $t_1 = -1$ and $t_2 \in \{-5, 6\}$. We have the solution $(\nu_{2a},\nu_{2b},\nu_{11a}) = (0,1,0)$. After considering the additional restriction $\mu_1(u,\chi_5,*) = \frac{1}{22}(7\nu_{2a} - 9\nu_{2b} + 240) = \frac{231}{22}$. Since $\mu_1(u,\chi_5,*)$ is not an integer, we obtain a contradiction, so in this case there is no solution.
(iii) $\chi(u^{11}) = -2\chi(2a) + \chi(2b)$. We get $t_1 = 5$ and $t_2 = 1$.
(iv) $\chi(u^{11}) = 2\chi(2a) - \chi(2b)$. In this case there is no solution for t_1.
(v) $\chi(u^{11}) = \chi(u^{11}) = 3\chi(2a) - 2\chi(2b)$. We get $t_1 = 1$ and $t_2 = -3$.
(vi) $\chi(u^{11}) = -\chi(2a) + 2\chi(2b)$. We get $t_1 = 2$ and $t_2 = -2$.

Finally, assume that $\chi(u^{11}) = 3\chi(2a) - 2\chi(2b)$. Put $t_1 = 7\nu_{2a} - \nu_{2b} + \nu_{11a}$ and $t_2 = 3\nu_{2a} - 5\nu_{2b} - \nu_{11a}$. Again, by (2) we obtain the system of inequalities

$$\begin{aligned} \mu_1(u,\chi_2,*) = \tfrac{1}{22}(t_1 - 1) \geq 0; \qquad \mu_{11}(u,\chi_2,*) = \tfrac{1}{22}(-10t_1+10) \geq 0; \\ \mu_{11}(u,\chi_3,*) = \tfrac{1}{22}(10t_2+74) \geq 0; \quad \mu_0(u,\chi_3,*) = \tfrac{1}{22}(-10t_2+36) \geq 0, \end{aligned}$$

with integral solution $(t_1,t_2) = (1,-3)$. Now we substitute the obtained values of t_1 and t_2 into the system of equations (4). Then we can conclude that it is impossible to find integer solution of (4) for ν_{2a}, ν_{2b} and ν_{11a}.

• Let u be a unit of order 33. By (1) and Proposition 2 we have that

$$\nu_{3a} + \nu_{3b} + \nu_{11a} = 1.$$

Since $|u^{11}| = 3$, for any character χ of G we need to consider six cases, defined by part (iv) of the Theorem. Put

$$(\alpha,\beta) = \begin{cases} (55,55), & \text{if} \quad \chi(u^{11}) = \chi(3a); \\ (61,52), & \text{if} \quad \chi(u^{11}) = \chi(3b); \\ (49,58), & \text{if} \quad \chi(u^{11}) = 2\chi(3a)-\chi(3b); \\ (43,61), & \text{if} \quad \chi(u^{11}) = 3\chi(3a)-2\chi(3b); \\ (67,49), & \text{if} \quad \chi(u^{11}) = -\chi(3a)+2\chi(3b); \\ (37,64), & \text{if} \quad \chi(u^{11}) = 4\chi(3a)-3\chi(3b). \end{cases} \tag{8}$$

By (2) we obtain the system of inequalities

$$\begin{aligned} \mu_0(u,\chi_3,*) = \tfrac{1}{33}(20(3\nu_{3b}+\nu_{11a})+\alpha) \geq 0; \\ \mu_{11}(u,\chi_3,*) = \tfrac{1}{33}(-10(3\nu_{3b}+\nu_{11a})+\beta) \geq 0, \end{aligned}$$

which has no integer solutions in any of the six cases of (8).

• Let u be a unit of order 35. By (1) and Proposition 2 we get $\nu_{5a}+\nu_{7a}+\nu_{7b}=1$. Since $|u^5|=7$, we need to consider four cases for any character χ of G. They are defined by part (v) of the Theorem. By (2), in all of the cases we get the system

$$\mu_0(u,\chi_2,*)=\tfrac{1}{35}(24(3\nu_{5a}+2\nu_{7a}+2\nu_{7b})+47)\geq 0;$$
$$\mu_7(u,\chi_2,*)=\tfrac{1}{35}(-6(3\nu_{5a}+2\nu_{7a}+2\nu_{7b})+32)\geq 0,$$

which has no integer solutions.

• Let u be a unit of order 46. By (1) and Proposition 2 we have that

$$\nu_{2a}+\nu_{2b}+\nu_{23a}+\nu_{23b}=1.$$

Put $\alpha=\begin{cases}-3, & \text{if } \chi(u^{23})=\chi(2a);\\ 1, & \text{if } \chi(u^{23})=\chi(2b);\\ 9, & \text{if } \chi(u^{23})=-2\chi(2a)+3\chi(2b);\\ -7, & \text{if } \chi(u^{23})=2\chi(2a)-\chi(2b);\\ 5, & \text{if } \chi(u^{23})=-\chi(2a)+2\chi(2b).\end{cases}$

According to (2) we obtain that

$$\mu_0(u,\chi_2,3)=-\mu_{23}(u,\chi_2,3)=$$
$$=\tfrac{1}{46}(22(6\nu_{2a}-2\nu_{2b}-\nu_{23a}-\nu_{23b})+\alpha)=0,$$

which is impossible.

Now let $\chi(u^{23})=3\chi(2a)-2\chi(2b)$. Put $t_1=3\nu_{2a}-5\nu_{2b}+\nu_{23a}+\nu_{23b}$, then by (2) we obtain the system of inequalities

$$\mu_{23}(u,\chi_3,*)=\tfrac{1}{46}(22t_1+42)\geq 0;\quad \mu_0(u,\chi_3,*)=\tfrac{1}{46}(-22t_1+4)\geq 0,$$

which has no solution for t_1.

• Let u be a unit of order 55. By (1) and Proposition 2 we have that $\nu_{5a}+\nu_{11a}=1$. By (2) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{55}(40(3\nu_{5a}+\nu_{11a})+45)\geq 0;$$
$$\mu_{11}(u,\chi_2,*)=\tfrac{1}{55}(-10(3\nu_{5a}+\nu_{11a})+30)\geq 0;$$
$$\mu_1(u,\chi_2,*)=\tfrac{1}{55}(3\nu_{5a}+\nu_{11a}+19)\geq 0.$$

It easy to check that last system of inequalities has no integral solution.

• Let u be a unit of order 69. By (1) and Proposition 2 we have that

$$\nu_{3a}+\nu_{3b}+\nu_{23a}+\nu_{23b}=1.$$

Since $|u^{23}|=3$ and by part (iv) of the Theorem we have six cases for units of order 3, and, furthermore, $\chi(u^3)\in\{\chi(23a),\chi(23b)\}$, for any character χ of G we need to consider 12 cases. Put

$$(\alpha,\beta)=\begin{cases}(23,23), & \text{if } \chi(u^{23})=\chi(3a);\\ (29,20), & \text{if } \chi(u^{23})=\chi(3b);\\ (17,26), & \text{if } \chi(u^{23})=2\chi(3a)-\chi(3b);\\ (11,29), & \text{if } \chi(u^{23})=3\chi(3a)-2\chi(3b);\\ (35,17), & \text{if } \chi(u^{23})=-\chi(3a)+2\chi(3b);\\ (5,32), & \text{if } \chi(u^{23})=4\chi(3a)-3\chi(3b).\end{cases} \tag{9}$$

By (2) in all of the 12 cases we obtain the system

$$\mu_0(u,\chi_3,*)=\tfrac{1}{69}(44(3\nu_{3b}-\nu_{23a}-\nu_{23b})+\alpha)\geq 0;$$
$$\mu_{23}(u,\chi_3,*)=\tfrac{1}{69}(-22(3\nu_{3b}-\nu_{23a}-\nu_{23b})+\beta)\geq 0,$$

which has no integral solutions.

• Let u be a unit of order 77. By (1) and Proposition 2 we have that

$$\nu_{7a}+\nu_{7b}+\nu_{11a}=1.$$

Since $|u^{11}| = 7$, we need to consider four cases for any character χ of G. They are defined by part (v) of the Theorem. By (2) we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{77}(60(2\nu_{7a} + 2\nu_{7b} + \nu_{11a}) + 45) \geq 0;$$
$$\mu_{11}(u,\chi_2,*) = \tfrac{1}{77}(-10(2\nu_{7a} + 2\nu_{7b} + \nu_{11a}) + 31) \geq 0,$$

which has no integral solutions.
• Let u be a unit of order 115. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{23a} + \nu_{23b} = 1.$$

Since $|u^5| = 23$ and $\chi(u^5) \in \{\chi(23a), \chi(23b)\}$, we need to consider two cases for any character χ of G. In both cases by (2) we get the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{115}(264\nu_{5a} + 35) \geq 0; \qquad \mu_{23}(u,\chi_2,*) = \tfrac{1}{115}(-66\nu_{5a} + 20) \geq 0,$$

which has no integral solution. The proof is done.
• Let u be a unit of order 161. By (1) and Proposition 2 we have that

$$\nu_{7a} + \nu_{7b} + \nu_{23a} + \nu_{23b} = 1.$$

Since $|u^{23}| = 7$ and $\chi(u^7) \in \{\chi(23a), \chi(23b)\}$, for any character χ of G we need to consider eight cases, defined by part (v) of the Theorem. By (2) in all eight cases we obtain the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{161}(264(\nu_{7a} + \nu_{7b}) + 35) \geq 0;$$
$$\mu_{23}(u,\chi_2,*) = \tfrac{1}{161}(-44(\nu_{7a} + \nu_{7b}) + 21) \geq 0,$$

which has no integral solution.
• Let u be a unit of order 253. By (1) and Proposition 2 we have that

$$\nu_{11a} + \nu_{23a} + \nu_{23b} = 1.$$

Since $\chi(u^{11}) \in \{\chi(23a), \chi(23b)\}$, we consider two cases for any character χ of G. Put $\quad t_1 = 11\nu_{23a} - 12\nu_{23b} \quad$ and $\quad \alpha = \begin{cases} 23 & \text{if } \chi(u^{11}) = \chi(23a); \\ 0 & \text{if } \chi(u^{11}) = \chi(23b). \end{cases}$
By (2) in both cases we obtain

$$\mu_0(u,\chi_2,*) = \tfrac{1}{253}(220\nu_{11a} + 33) \geq 0; \quad \mu_{23}(u,\chi_2,*) = \tfrac{1}{253}(-22\nu_{11a} + 22) \geq 0;$$
$$\mu_1(u,\chi_2,2) = \tfrac{1}{253}(t_1 + \alpha) \geq 0; \qquad \mu_{55}(u,\chi_2,2) = \tfrac{1}{253}(-10t_1 + \alpha) \geq 0,$$

so $\nu_{11a} = 1$ and $t_1 = 0$, so the solution is $(\nu_{11a}, \nu_{23a}, \nu_{23b}) = (1, 0, 0)$. Now we compute that $\mu_1(u,\chi_2,*) = \frac{1}{253}(\nu_{11a} + 22) = \frac{23}{253}$ is not an integer, thus, there is no solution in this case.

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Appendix

Possible partial augmentations $(\nu_{3a}, \nu_{3b}, \nu_{5a}, \nu_{15a}, \nu_{15b})$ for units of order 15:

$(-3,0,5,-1,0)$,	$(-3,0,5,0,-1)$,	$(-2,-1,5,-1,0)$,	$(-2,-1,5,0,-1)$,
$(-2,2,5,-2,-2)$,	$(-1,1,5,-3,-1)$,	$(-1,1,5,-2,-2)$,	$(-1,1,5,-1,-3)$,
$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1})$,	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0})$,	$(0,3,0,-1,-1)$,	$(1,-1,0,0,1)$,
$(1,-1,0,1,0)$,	$(1,2,0,-2,0)$,	$(1,2,0,-1,-1)$,	$(1,2,0,0,-2)$,
$(2,1,0,-2,0)$,	$(2,1,0,-1,-1)$,	$(2,1,0,0,-2)$,	$(2,4,0,-3,-2)$,
$(2,4,0,-2,-3)$,	$(3,0,-5,1,2)$,	$(3,0,-5,2,1)$,	$(3,3,-5,0,0)$,
$(4,-1,-5,1,2)$,	$(4,-1,-5,2,1)$,	$(4,2,-5,-1,1)$,	$(4,2,-5,0,0)$,
$(4,2,-5,1,-1)$,	$(5,1,-5,-1,1)$,	$(5,1,-5,0,0)$,	$(5,1,-5,1,-1)$,
$(5,4,-5,-2,-1)$,	$(5,4,-5,-1,-2)$.		

Possible partial augmentations $(\nu_{3a}, \nu_{3b}, \nu_{7a}, \nu_{7b}, \nu_{21a}, \nu_{21b})$ for units of order 21:

$$
\begin{array}{lll}
(0,0,-3,3,-1,2), & (0,0,-2,2,0,1), & (0,0,-1,1,0,1), \\
(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1}), & (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{0}), & (0,0,1,-1,1,0), \\
(0,0,2,-2,1,0), & (0,0,2,-2,2,-1), & (0,0,3,-3,2,-1), \\
(1,2,-2,2,-2,0), & (1,2,-1,1,-2,0), & (1,2,-1,1,-1,-1), \\
(1,2,0,0,-1,-1), & (1,2,1,-1,-1,-1), & (1,2,1,-1,0,-2), \\
(1,2,2,-2,0,-2), & (4,2,-4,-3,0,2), & (4,2,-4,-3,1,1), \\
(4,2,-3,-4,1,1), & (4,2,-3,-4,2,0), & (0,0,-2,2,-1,2).
\end{array}
$$

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